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THE USE OF B-SPLINES IN THE ESTIMATION OF POWER SPECTRA, (U)
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THE USE OF B-SPLINES IN THE ESTIMATION
OF POWER SPECTRA

J. Rodney Grisham and James R. Thompson

Abstract

A class of data windows is proposed for use in the estimation of the power spectra of stationary stochastic processes. These windows, which are generalizations of the standard Parzen filter, are constructed by the use of appropriately normalized B-splines. It is demonstrated how this window class may be computationally implemented using a Fast Fourier Transform algorithm. The efficiency of the resulting procedure is generally a significant improvement over the state of the art with little additional computer time required.

1. INTRODUCTION

This paper is concerned with the problem of digitally estimating the power spectrum of a wide-sense stationary, ergodic, stochastic process from a sample function of finite length. Not long after Cooley and Tukey [5] introduced the fast Fourier transform algorithm (FFT), its application to the estimation of power spectra was discussed by Bingham, Godfrey, and Tukey [2] and by Welch [18]. They suggested a method based on complex demodulation and pointed out the computational speed advantages which can be obtained by using the FFT for computing Fourier periodograms. Welch proposed a direct method which has become the standard numerical approach to the problem of power spectral estimation. This procedure has three parts: subdividing the available data, using a data window to compute a smoothed spectral estimate for each segment, and averaging these smoothed spectral estimates. The present paper proposes a class of data windows derived from the B-spline basis functions of Schoenberg [14]. Empirically,

the result of using these data windows is a significant improvement in estimates of power spectra without a noticeable increase in the computational effort. We have chosen to use continuous notation throughout; conversion to discrete formulae where appropriate is straightforward.

2. PRELIMINARIES

We first outline some basic notation and results which are assumed. Here we follow Stein and Weiss [16]. L^1 denotes the Banach space of Lebesgue measurable functions which are (Lebesgue) integrable over the real line \mathbb{R} . For $f \in L^1$, the Fourier transform of f is the function defined by

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(t) e^{-2\pi i \xi t} dt$$

for all $\xi \in \mathbb{R}$ where $i = \sqrt{-1}$. The notation $(\dots)^{\wedge}$ denotes the Fourier transform of (\dots) . For $f, g \in L^1$, the convolution $h = f * g$ is defined as the function $h \in L^1$ given by

$$h(x) = \int_{-\infty}^{\infty} f(t) g(x-t) dt$$

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$$h(\tau) = \int_{-\infty}^{\infty} f(\tau-t)g(t)dt$$

for all $\tau \in \mathbb{R}$. If $f, g \in L^1$, then $(f * g)^{\wedge} = \hat{f}\hat{g}$. Similarly, $(fg)^{\wedge} = \hat{f}\hat{g}$ provided the functions involved are such that all operations make sense. For $a > 0$, δ_a represents a dilation operator, i.e., $\delta_a f(t) = f(at)$. Whenever $f, g \in L^1$, it is clear that

$$(\delta_a f * \delta_a g)(\tau) = \frac{1}{a} \delta_a [(f * g)(\tau/a)]$$

$$\text{and } (\delta_a f)^{\wedge}(\xi) = \frac{1}{a} \hat{f}(\xi/a).$$

We shall consider a sample function $x(t)$ on a finite interval, $-\frac{T}{2} \leq t \leq \frac{T}{2}$ unless otherwise stated, of a stochastic process $\{X(t); -\infty < t < \infty\}$. We assume this underlying stochastic process to be wide-sense stationary (Doob[8]) and ergodic. For simplicity we take the mean value to be zero. Let $X(t)$ have the spectral density function $P(f)$. The terminology related to spectral analysis used in this paper is generally that of Blackman and Tukey [3].

It is apparent that high resolution and stability (small bias and variance) are desired qualities of power spectral estimates. However, a compromise between the two must be made. In order to obtain a stable spectral estimate using the classical indirect method, an appropriate function of lag must modify the sample autocovariance function. Fundamentally the indirect method estimates $P(f)$ via

$$P_{i.m.}(f) = \int_{-\tau}^{\tau} C(\tau)L(\tau)e^{-2\pi i f \tau} d\tau$$

where $L(\tau)$ is the finitely supported lag window and $C(\tau)$ is the sample autocovariance function

$$C(\tau) = \frac{1}{T} \int_{-(T-|\tau|)/2}^{(T-|\tau|)/2} x(t + \frac{\tau}{2})x(t - \frac{\tau}{2})dt.$$

Usually the support of $L(\tau)$ is much less than the entire interval $-T \leq \tau \leq T$.

The arguments for using lag windows (e.g., see Jenkins and Watts [10]) can be adapted to show that a procedure which utilizes the FFT to estimate power spectra directly should incorporate a data window. That is, for a direct procedure to have a proper balance between resolution and stability, the data itself must be modified by a suitable function of time. Thus, basically the

FFT is used to evaluate

$$P_{d.m.}(f) = \frac{1}{T} \left| \int_{-T/2}^{T/2} W(t)x(t)e^{-2\pi i f t} dt \right|^2$$

at discrete frequencies where $W(t)$ is the finitely supported data window.

The end result of either of these modifications is that we in fact estimate smoothed values of $P(f)$. More specifically, if we let $P_o(f)$ represent the estimate computed by either the indirect or the direct approach, we have

$$\text{avg}[P_o(f)] \approx Q(f) * P(f)$$

where the average may be taken either over the ensemble or along time [3]. $Q(f)$ is called the spectral window corresponding to the window used in the computation.

The direct and indirect approaches to estimating the power spectrum of a stochastic process are not fundamentally different. Blackman and Tukey [3] have derived the connection between a data window $W(t)$ and its "equivalent" lag window $L(\tau)$. This relationship turns out to be the correlation integral

$$L(\tau) = \frac{1}{T} \int_{-\infty}^{\infty} W(t)W(t+\tau)dt.$$

By "equivalent" we mean that the appropriate use of the respective "equivalent" windows yields power spectral estimates which are equal in expectation. Note that if the data window is an even function of time, then the equivalent lag window is proportional to the convolution of the data window with itself.

At this point it should be clear that in the indirect method, the spectral window $Q(f)$ corresponding to the lag window $L(\tau)$ is simply the Fourier transform of $L(\tau)$. For the direct method it then follows that the spectral window $Q(f)$ corresponding to the data window $W(t)$ is given by $Q(f) = \frac{1}{T} |J(f)|^2$ where $J(f)$ is defined to be the Fourier transform of $W(t)$. $J(f)$ is called the frequency window corresponding to $W(t)$. We again refer the reader to Blackman and Tukey [3].

3. SPLINE DATA WINDOWS

3.1 MOTIVATION

In light of the rather simple relationship between a data window and its equivalent lag window, one should not ignore the extensive effort which has been put into the construction of lag windows by numerous authors nor the properties of those windows. See for example the compilation given by Jenkins [9] or by Parzen [12,13].

For instance, recall the continuity class to which these functions belong. In particular we observe that one of the lag windows which has been proposed by Parzen [13] is proportional to a dilated, fourth order B-spline basis function. One of the two data windows suggested by Welch [18] has the shape of a second order B-spline basis function; he recognized that his proposed window yields the Parzen spectral window. Henceforth we will refer to this window which is given by

$$W(t) = \begin{cases} \sqrt{3} \left(1 - \frac{|t|}{T/2}\right), & |t| \leq T/2 \\ 0, & \text{otherwise} \end{cases}$$

as the Parzen data window, even though it is also a member of the family of spline windows proposed in this paper. Today, Welch's procedure (described in the introduction) incorporating the Parzen window is undoubtedly the most widely used method of direct power spectral estimation. Bingham et al [2] proposed the following data window which is continuous through the first derivative:

$$W(t) = \begin{cases} 2\sqrt{\frac{2}{7}}, & |t| \leq .4T \\ \sqrt{\frac{2}{7}} \left[1 - \cos\left(\frac{5\pi}{T}(T-2|t|)\right)\right], & .4T \leq |t| \leq .5T \\ 0, & \text{otherwise} \end{cases}$$

We have normalized this window properly although this was not done in [2].

From another point of view, we note that under the Fourier transform operator the effect of an occurrence at one point in time tends to be spread over all frequencies. Furthermore, time functions with corners have Fourier transforms that have a "ringing" effect. Thus it seems desirable to have a data window of a high continuity

class so that the modified sample function goes smoothly to zero at its ends.

Such considerations motivate one to examine the possibility of using data windows based upon higher order B-spline basis functions. A preliminary survey is encouraging. First, we may choose the desired continuity class. Secondly, the corresponding spectral windows have the shape $\left(\frac{\sin u}{u}\right)^{k+1}$ so that the magnitude of the side lobes rapidly decreases relative to the main peak as the continuity (or order k) increases. In addition, all spectral windows are of non-negative type (Parzen [13]). Thirdly, deBoor [7] has derived a general algorithm which can be used for accurately evaluating any order B-spline basis function with any interval of support and at any mesh length.

3.2 THE FUNDAMENTAL SPLINE FUNCTIONS

Schoenberg [14] introduced the spline functions in 1946 with the following definition. With k a positive integer, a real function $\mathcal{A}_k(t)$ defined for all $t \in \mathbb{R}$ is called a spline function of order k or degree $k-1$ if it has the following properties:

- (i) $\mathcal{A}_k(t)$ is of class $C^{k-2}(\mathbb{R})$.
- (ii) $\mathcal{A}_k(t)$ is composed of polynomial arcs of degree at most $k-1$.
- (iii) The polynomial arcs are joined nowhere other than possibly at integers n if k is even or at points $n + \frac{1}{2}$ if k is odd. The spline is said to have knots at these points. If we define

$$y_+^{k-1} = \begin{cases} y^{k-1}, & y \geq 0 \\ 0, & y \leq 0 \end{cases}$$

with exception of the special case $0_+^{1-1} = \frac{1}{2}$ and let δ^k represent the k^{th} order central difference operator with unity step, i.e., $\delta^k f(y) =$

$$f(y) \text{ and } \delta^k f(y) = \delta^{k-1} f(y + \frac{1}{2}) - \delta^{k-1} f(y - \frac{1}{2})$$

for $k = 1, 2, \dots$, then the function

$$M_k(t) = \frac{1}{(k-1)!} \delta^{k-1} t_+^{k-1}$$

is easily shown to be a simple example of a spline function of order k . $M_k(t)$ is called the funda-

mental B-spline basis function of order k . We list a few of the many important properties of $M_k(t)$ and refer the reader to Schoenberg [14,15], Curry and Schoenberg [6], and to deBoor [7] for these and other properties. $M_k(t)$ is positive in the open interval $(-\frac{k}{2}, \frac{k}{2})$ and identically zero elsewhere; it has knots at $-\frac{k}{2} + v$ where v is an integer satisfying $0 \leq v \leq k$. $M_k(t)$ is an even function, and its integral over R is unity. $M_k(t)$ also has the representation

$$M_k(t) = \frac{1}{(k-1)!} \sum_{v=0}^k (-1)^{k-v} \binom{k}{v} \left(-\frac{k}{2} + v - t\right)_+^{k-1}.$$

For example, the fourth order function is given by

$$M_4(t) = \begin{cases} \frac{1}{6}(2 - |t|)^3 - \frac{2}{3}(1 - |t|)^3, & |t| \leq 1 \\ \frac{1}{6}(2 - |t|)^3, & 1 \leq |t| \leq 2 \\ 0, & \text{otherwise.} \end{cases}$$

We shall require two integral properties:

$$\hat{M}_k(f) = \left(\frac{\sin \pi f}{\pi f}\right)^k$$

and

$$(M_p * M_k)(\tau) = M_{p+k}(\tau).$$

3.3 THE PROPOSED CLASS OF DATA WINDOWS

3.3.1 Introduction.

The class of spline data windows we propose consists of the fundamental spline functions each dilated so that its support is equal to the length of the sample function and normalized so that the total power of the data remains invariant when the spline data windows are applied. If we let C_k represent the normalization factor, then we can use the results of the previous sections to obtain the following table of the spline data windows and of the corresponding frequency windows, lag windows, and spectral windows.

Since the total power σ^2 is given by

$$\sigma^2 = \int_{-T/2}^{T/2} x^2(t) dt = T \text{Var}[x],$$

we seek constants C_k depending upon the order k of the spline data window such that also

$$\sigma^2 = \int_{-T/2}^{T/2} [C_k W_k(t) x(t)]^2 dt$$

where we use the tilde to indicate an un-normalized window. The expectation of this equation is

$$\begin{aligned} \sigma^2 &= C_k^2 \int_{-T/2}^{T/2} E[\tilde{W}_k(t)] E[x^2(t)] dt \\ &= C_k^2 \text{Var}[x] \int_{-T/2}^{T/2} \tilde{W}_k^2(t) dt. \end{aligned}$$

Changing the variable of integration we obtain

$$C_k^2 = k \int_{-1/2}^{1/2} \tilde{M}_k^2(t) dt.$$

An application of Parseval's theorem yields

$$C_k^2 = \pi k / 2 \int_0^{\pi} \left(\frac{\sin u}{u}\right)^{2k} du.$$

Therefore, we can calculate the appropriate normalizing factor for any order spline data window:

$C_1 = 1.0$ and

$$C_k = \left[2 \sum_{i=0}^{k-1} \frac{(-1)^i (k-i)! (k-1)!}{i! (2k-i-1)!} \right]^{-1/2} \text{ for } k \geq 2.$$

Normalizing for invariance of total power is equivalent to the requirement

$$1(0) = \int_{-\infty}^{\infty} Q(f) df = 1.$$

Data Window $W_k(t)$	$C_k \frac{1}{\sqrt{T}} M_k(t) = C_k M_k\left(\frac{kt}{T}\right)$
Frequency Window $J_k(f)$	$C_k \left(\frac{1}{k}\right) \frac{1}{T/k} \left[\left(\frac{\sin(\pi f T/k)}{\pi f T/k} \right)^k \right] = \frac{C_k}{k} \left(\frac{\sin(\pi f T/k)}{\pi f T/k} \right)^k$
Equivalent Lag Window $L_k(\tau)$	$\frac{C_k^2}{T} \left[\frac{T}{k} \delta_{W_T} M_{2k}(\tau) \right] = \frac{C_k^2}{k} M_{2k}\left(\frac{k\tau}{T}\right)$
Spectral Window $Q_k(f)$	$\left(\frac{C_k}{k} \right) \left(\frac{T}{k} \right) \frac{1}{T/k} \left[\left(\frac{\sin(\pi f T/k)}{\pi f T/k} \right)^{2k} \right] = \frac{C_k^2 T}{k^2} \left(\frac{\sin(\pi f T/k)}{\pi f T/k} \right)^{2k}$

TABLE 1.

3.3.2 Resolution and Stability Considerations

The convergence properties of the spline windows are rather simple. Clearly for any k

$$\lim_{T \rightarrow \infty} Q_k(f) = \lim_{T \rightarrow \infty} \frac{C_k T}{k} \left(\frac{\sin(\pi f T/k)}{\pi f T/k} \right)^{2k}$$

$$\begin{cases} 0, & f \neq 0 \\ \infty, & f = 0 \end{cases}$$

and $\int_{-\infty}^{\infty} Q_k(f) df = 1$ for all T .

Thus, as the extent of the sample function is increased, the spectral window converges to the Dirac delta function (distribution), and therefore the data window is converging to the constant function identically equal to unity.

Figure 1 illustrates some of the spline data windows; the corresponding spectral windows are shown in Figure 2. In Figure 3 the spectral windows have been plotted on a logarithmic scale in an attempt to represent the extreme differences more clearly.

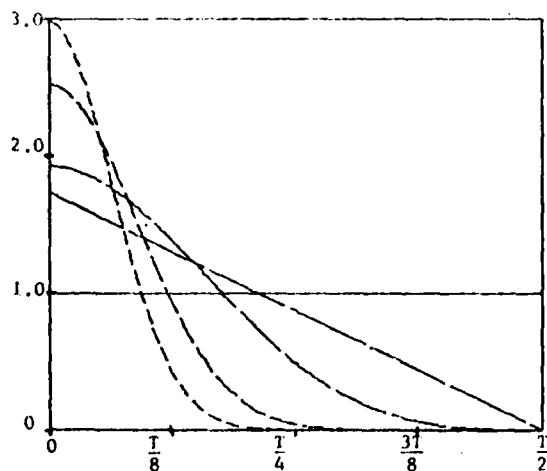


Figure 1. Data Windows

W_1 _____
 W_2 _____
 W_4 _____
 W_{11} _____
 W_{21} _____

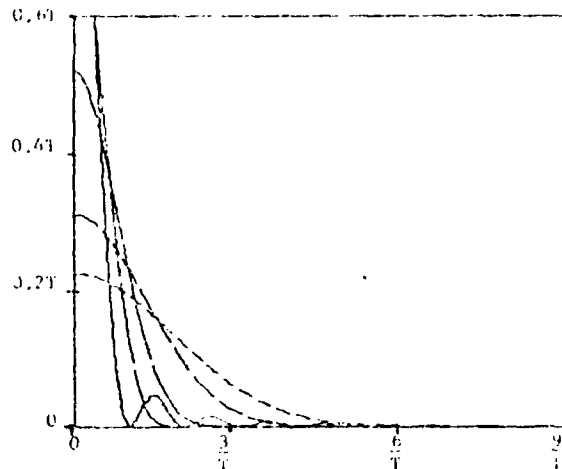


Figure 2. Spectral Windows

Q_1 _____
 Q_2 _____
 Q_4 _____
 Q_{11} _____
 Q_{21} _____

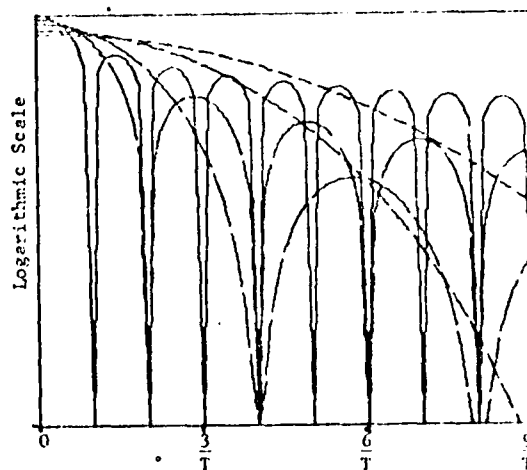


Figure 3. Spectral Windows as in Fig. 2.

We point out that W_1 is simply a truncation function, L_1 is the lag window suggested by Bartlett [1], and Q_1 is the corresponding Bartlett spectral window. W_2 , L_2 , and Q_2 are respectively the Parzen data, lag, and spectral windows.

For the problem of balancing the resolution and stability of power spectral estimates, explicit expressions for the bias and variance of smoothed estimators have been developed in the literature. Jenkins and Watts [10] and others give these expressions for the indirect computing method. Welch [17,18] derives the expected value and variance of the discrete, directly computed estimates for a stochastic process which is Gaussian, in addition to our previous assumptions about the process. In each case some property of the window being used is explicitly involved. The practitioner, however, needs some device to aid in the selection of a window that will yield a compromise suitable for his purposes. The most common measure of windows is called bandwidth (by analogy with filtering problems).

Several definitions are used for bandwidth. Blackman and Tukey [3] use instead the term "equivalent width, which they define to be the ratio of the square of the integral of the spectral window to the integral of its square. If we define

$$I = \int_{-\infty}^{\infty} Q^2(f) df,$$

then the equivalent width is given by

$$B_1(Q) = \frac{1}{I} \left[\int_{-\infty}^{\infty} Q(f) df \right]^2.$$

Jenkins and Watts [10] define bandwidth to be the width of the rectangular (spectral) window for which the variance of the indirectly computed spectral estimate is the same as it is for the given window. This definition of bandwidth yields [10]

$$B_1(Q) = \frac{1}{I}.$$

As has been noted, for a properly normalized window $\int_{-\infty}^{\infty} Q(f) df = 1$; therefore $B_1 = B_1$. We

mention B_1 for its application to the spline lag windows given in Table 1. However, its defining property is no longer true for the direct method of computation which we advocate. For the class of spline windows we have

$$\begin{aligned} I_k &= \int_{-\infty}^{\infty} Q_k^2(f) df \\ &= \frac{2C_k^2 T}{\pi k^3} \int_0^{\pi} \left(\frac{\sin u}{u} \right)^{4k} du \\ &= \frac{4C_k^2 T}{k^3} \sum_{i=0}^{k-1} \frac{(-1)^i (2k-i)^{2(2k-1)}}{i!(4k-i)!}. \end{aligned}$$

Thus $B_1(Q_k)$ is easily determined by inverting I_k . By using least-squares techniques for $k = 3, \dots, 34$, we obtain the approximation

$$B_1(Q_k) \approx (1.68(k)^{0.45} - 0.52)/T.$$

Parzen [13] defines bandwidth as the width of a rectangle which has the same area and same maximum height as the given spectral window. Therefore

$$B_2(Q) = \frac{\int_{-\infty}^{\infty} Q(f) df}{\max_f Q(f)}$$

and for the spline windows we have

$$B_2(Q_k) = \frac{k^2}{C_k^2 T}.$$

Another commonly used measure of bandwidth is the distance between the half-power points of the spectral window. Thus, for an even spectral window centered at $f = 0$, $B_3(Q) = 2\tilde{f}$ where \tilde{f} is the first frequency such that $Q(\tilde{f}) = \frac{1}{2}Q(0)$. Let $j_k[\cdot]$ represent the spherical Bessel functions of the first kind, then

$$B_3(Q_k) = \frac{2k}{C_k} j_{2k}^{-1} \left[\left(\frac{1}{2} \right)^{1/4k} \right].$$

Approximations can be precisely determined with reference to tables (e.g., [11]). Also, frequently references are made to the separation between the first zeros of spectral windows. The zeros of $Q_k(f)$ occur at $f = \frac{n}{C_k T}$ where n is a natural number.

Various authors have shown that the bias of smoothed spectral estimators increases as the bandwidth of the window increases, but that the variance is inversely proportional to the band-

length. It can also be argued that the bandwidth of the window must be at least as small as the width of the narrowest important detail in the spectrum.

Table 2 is a compilation of the various constants associated with the spline windows given in Table 1 with order up to $k=21$.

Order k	Constant C_k	Bandwidths $\times T$		
		$FB_1(Q_k)$	$FB_2(Q_k)$	$FB_3(Q_k)$
1	1.0000	1.5000	1.0000	0.9859
2	1.7320	1.8543	1.3333	1.2757
3	2.3355	2.3037	1.6660	1.5716
4	2.8887	2.6857	1.9175	1.8201
5	3.4063	3.0203	2.1521	2.0385
6	3.9327	3.3214	2.3636	2.2257
7	4.3770	3.5974	2.5576	2.4168
8	4.8348	3.8536	2.7379	2.5853
9	5.2785	4.0939	2.9071	2.7334
10	5.7102	4.3209	3.0669	2.8930
11	6.1312	4.5365	3.2188	3.0351
12	6.5427	4.7423	3.3639	3.1704
13	6.9458	4.9396	3.5030	3.3011
14	7.3413	5.1292	3.6367	3.4264
15	7.7298	5.3121	3.7657	3.5473
16	8.1118	5.4890	3.8904	3.6641
17	8.4880	5.6602	4.0113	3.7774
18	8.8587	5.8265	4.1286	3.8873
19	9.2243	5.9882	4.2427	3.9943
20	9.5851	6.1456	4.3538	4.0984
21	9.9414	6.2990	4.4621	4.2000

TABLE 2.

3.4 COMPUTING WITH SPLINE DATA WINDOWS

Except for the choice of data windows, the method we propose for computing power spectra is that of Welch [18]. Departing from our convention, we assume that we now have a sample function $x(t)$ over the interval $0 \leq t \leq S$. Using for example a bandwidth criterion, choose the order k of the spline window to be used and then the window length T . Subdivide $x(t)$ into

N segments each of length T and with a displacement D between the initial points of consecutive segments. Denote the segments by $x_j(t)$. Thus $(N-1)D + T = S$, and

$$x_j(t) = x\left(t + (j-1)D + \frac{T}{2}\right) \text{ for } j=1, \dots, N; -\frac{T}{2} \leq t \leq \frac{T}{2}.$$

Welch uses the choice of D only as a parameter to increase the stability of the resulting estimated power spectrum. In the case of spline windows an additional consideration should be made. If $D=T$, some of the data is weighted substantially heavier than the remainder. As can be seen from Figure 1, the inequity is more pronounced as the order of the spline window increases in the sense that the spline data windows become progressively more peaked. Since one normally assumes that all portions of his data are equally representative of the stochastic process, intuitively the order of the spline window should also influence D . In practice a good rule of thumb for relatively short sample functions seems to be $D=T$ for $k=4$, $D=\frac{3}{6}T$ for $k=10$, and $D=\frac{1}{4}T$ for $k=20$ with the overlap for other order windows based on these. Next compute the smoothed spectral estimates

$$P_j(f) = \frac{1}{T} \left| \int_{-T/2}^{T/2} x_j(t) W_k(t) e^{-2\pi i f t} dt \right|^2, \quad j=1, \dots, N.$$

Finally, the estimated power spectrum is given by

$$P_e(f) = \frac{1}{N} \sum_{j=1}^N P_j(f).$$

Regardless of the order of spline window used, $P_e(f)$ is obviously a consistent estimator of $P(f)$ (at least) whenever the stochastic process is Gaussian. To see this, the expressions for $E[P_e(f)]$ and $\text{Var}[P_e(f)]$ as given by Welch [17,18] need only be compared with the convergence properties of the spline windows as discussed in this paper. That is, as $S \rightarrow \infty$ we have the following type convergence for a stationary Gaussian process: if $T \rightarrow \infty$ such that $\frac{T}{S} \rightarrow 0$, then $E[(P(f) - P_e(f))^2] \rightarrow 0$.

In implementing this procedure on a digital computer, one could derive a closed form for the B-spline basis functions like that given pre-

viously for $M_4(t)$. However, for a general program and one that would undoubtedly be numerically preferable, the algorithm of deBoor [7] is recommended for evaluating the spline functions. Of course, by symmetry only half the window need be directly evaluated. Brigham [4] thoroughly discusses the use of the FFT for computing the discrete Fourier transform which approximates the Fourier integral we used in writing $P_j(f)$.

4. EXPERIMENTAL RESULTS

The effectiveness of the spline data windows is readily apparent from experiments with machine generated autoregressive processes. Power spectral estimates computed using various order spline data windows are compared with those which used the Bingham-Godfrey-Tukey and Parzen data windows and with estimates computed by the indirect method of Blackman and Tukey incorporating the Hanning lag window [3]. The estimates were normalized by dividing out the variance of the stochastic process; as is customary when one-sided power spectra are used, the magnitude has been doubled.

For these experiments we used sample functions of autoregressive processes which were machine generated according to

$$x_t = \sum_{i=0}^m \alpha_i x_{t-i} + \varepsilon_t, \quad t = 1, \dots, 1024$$

where m is the order of the process, $\{\alpha_i\}$ are the regression coefficients, and ε_t is a white noise process with $E[\varepsilon_t] = 0$ and $\text{Var}[\varepsilon_t] = 1$. The particular processes which were used in the illustrations had the following coefficients:

- (i) First order, $\alpha_1 = -0.40$
- (ii) Second order, $\alpha_1 = 1.09, \alpha_2 = -0.50$
- (iii) Fourth order, $\alpha_1 = -0.25, \alpha_2 = 0.50, \alpha_3 = -0.33, \alpha_4 = -0.50$

the following scheme identifies the windows involved in the computation of the various power spectral estimates.

- (i) Hanning xx: estimate computed by the indirect method with the Hanning lag window [3] and using xx lags of the sample auto-

covariance function.

- (ii) B-G-T xx: Bingham-Godfrey-Tukey data window was used with segments of the sample function of length $T = xx$ samples.
- (iii) Parzen xx: Parzen data window was used with segments of the sample function of length $T = xx$ samples.
- (iv) Spline xx-yy-zz: Spline data window of order $k = xx$ was used with segments of the sample function each with length $T = yy$ samples and displacement $D = zz$ samples between initial points.

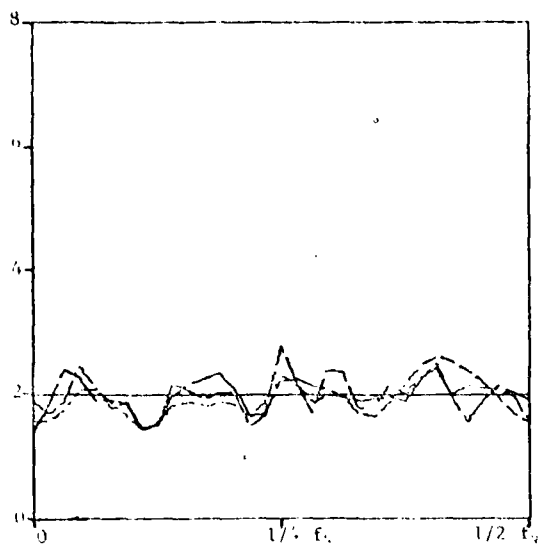
The graphs illustrating these experiments are found in Appendix 1.

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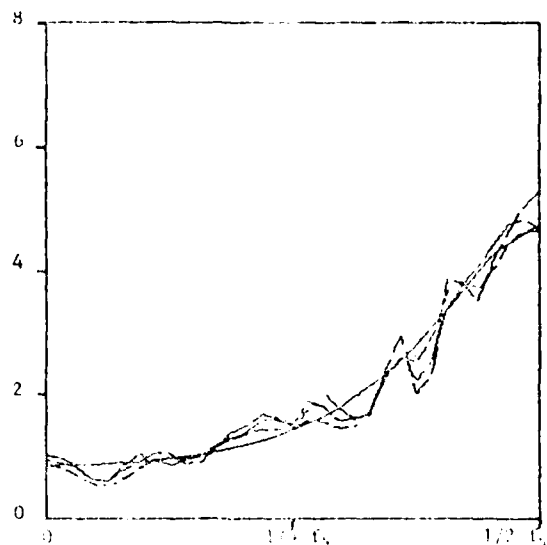
APPENDIX 1

White Noise Process

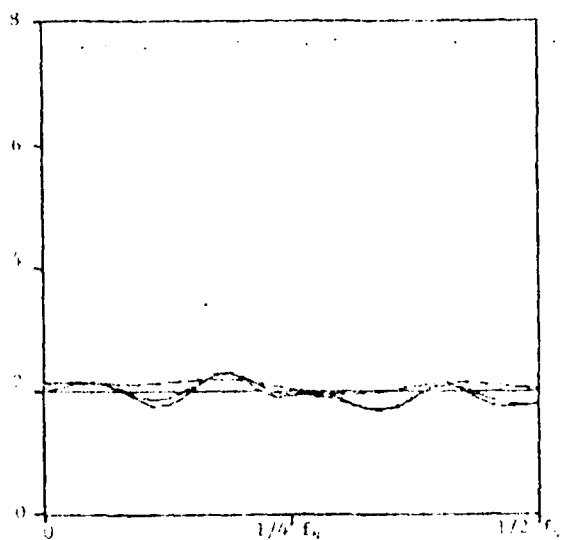


Theoretical ———
 Hanning 32 ———
 B-G-F 32 ———
 Parzen 32 ———

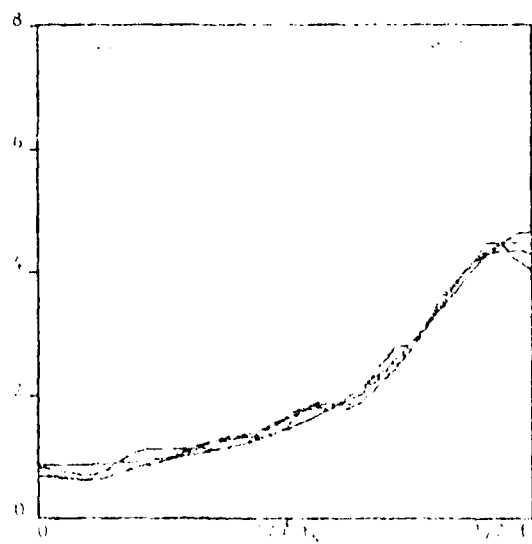
First Order Autoregressive Process



Theoretical ———
 Hanning 32 ———
 B-G-F 32 ———
 Parzen 32 ———

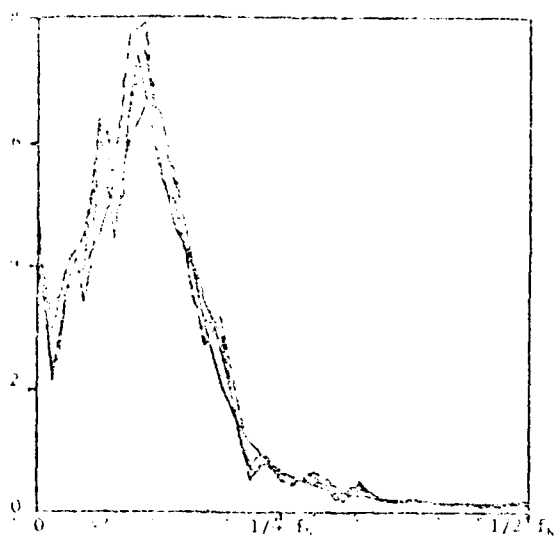


Theoretical ———
 Spline 4 32 16 ———
 Spline 6 32 16 ———
 Spline 21 32 16 ———

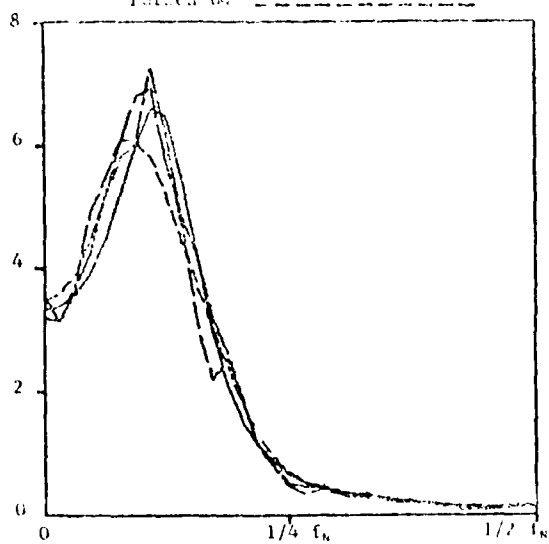


Theoretical ———
 Spline 4 32 16 ———
 Spline 8 64 16 ———
 Spline 21 64 16 ———

Second Order Autoregressive Process

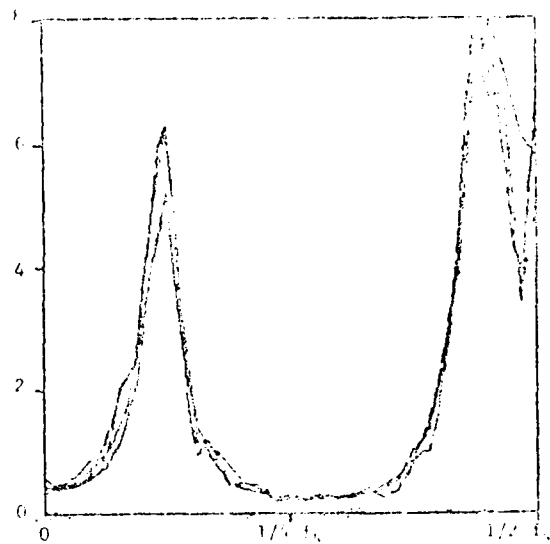


Theoretical —————
 Ranning 64 - - - - -
 B-G-T 64
 Parzen 64 -

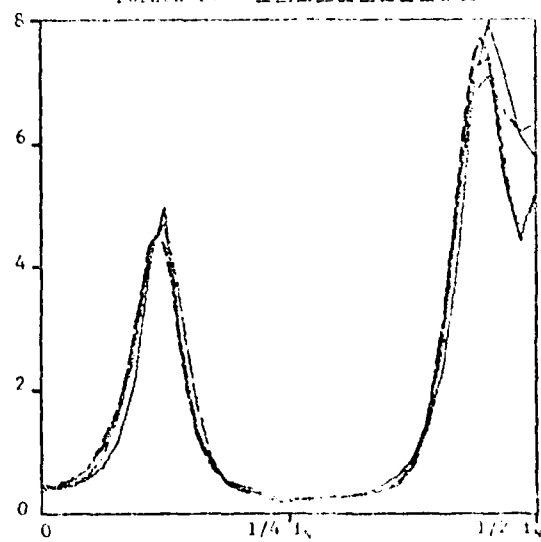


Theoretical —————
 Spline 3 64 48 - - - - -
 Spline 4 64 32
 Spline 21 64 16 -

Fourth Order Autoregressive Process



Theoretical —————
 Ranning 64 - - - - -
 B-G-T 64
 Parzen 64 -



Theoretical —————
 Spline 4 64 32 - - - - -
 Spline 8 64 24
 Spline 21 128 32 -

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